The Stability Analysis in a Continuous-Time Financial Market Model with Delay

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Abstract

In this paper we present a new continuous-time financial market model which is investigated to discuss the effect of time delay and market fraction of the fundamentalists on dynamics of asset prices. First, we analyze the global stability conditions of the fundamental steady state by constructing suitable Lyapunov functional. It is interesting to find the existence of the term of the feedback control, that is to say the new model has very good stability behaviours from the stable cycle or bifurcation. Second, numerical simulations show that the deterministic model is able to characterize long deviations of the market price from its fundamental price and stability behaviours observed in financial markets. It is the same to find that an increase in memory length not only can destabilize the market price, but also can stabilize an otherwise unstable market price, leading to stability switching as the memory length increases.

Keywords: Asset price, market behaviour, heterogeneous beliefs, time-delayed...

1. INTRODUCTION

We know that heterogeneous agent models (HAMs) have been developed to explain a range of market behaviours. But most of the HAMs in the literature are in discrete-time rather than continuous-time setup. In market analysis, Efficient Market Hypothesis (EMH) has been the hot topic of discussion in the academic finance literature [1], but the use of technical trading rules, such as moving average rules, still be widespread amongst financial market participants. Note that survey studies by Taylor and Allen [2], Frankel, J. A. [3] and Froot, K. A. et al. [4], in fact reveal that market participants use technical and fundamental analysis to assess financial markets. To examine the role of moving average rules in market stability theoretically, Chiarella et al.[5] proposes a discrete-time HAM in which demand for traded assets has both fundamentalist and chartist components. The chartist demand is governed by the difference between the current price and a moving average (MA). They show analytically and numerically that an increase in the lag length used in moving average can destabilize the market, leading to cyclic behavior of the market price around the fundamental price, and the MA plays a complicated role on the stability of financial markets. The discrete-time setup facilitates economic understanding and mathematical analysis, but it also faces some limitations when expectations of agents are formed in historical prices over different time periods. In particular, when dealing with MA rules in Chiarella et al. (2006), different lag lengths used in the MA rules lead to different dimensions of the system which need to be dealt with differently. However, a theoretical analysis on the impact of the memory length used in MA is difficult when the dimension of the system is high, which is the case as the memory length used in the MA becomes large. To overcome this difficulty, He et al.[6-7] proposes a heterogeneous agent model of financial markets in a continuous-time framework with time delay, which represents a memory length of a moving average rule in discrete-time HAMs, to study the impact of the memory length, but with the fundamental price as a content variable. Within a continuous-time framework. He et al. proposes a stochastic heterogeneous agent model of financial markets with time delays to unify various moving average rules used in discrete-time HAMs [7]. Intuitive conditions for the stability of the fundamental price of the deterministic model in terms of agents' behavior parameters and memory length have been obtained. We know clearly that the current market price always fluctuate around the fundamental values and the changes in the trend values is based on changes in the market prices change. The fundamentalist trade based on their estimated fundamental price. They believe that the market price is mean-reverting to the fundamental price that they can estimate based on various types of fundamental information, such as dividends, earnings, exports, general economic forecasts and so forth. The chartist trade based on charting signals generated from historical prices. They believe that the future market price follows a price trend. In any case, there are deterministic functional relations of both the market price and trend price given in He et al. [6-7]. In this paper we will discuss the relation between the fundamental value and the market supply and demand, find out the functional relation of the fundamental value, and build a new system. In the financial market, fundamental analysts or "fundamentalists" are grounded on fundamental analysis, the belief that present asset prices return to their fundamental values in the long run [8]. Buying undervalued and selling overvalued assets, as suggested by these rules, apparently has a stabilizing impact on market dynamics. Technical analysts or trend followers or "chartists", who think the assumption that prices tend to move in trends, and use various technical trading rules, attempt to forecast future prices by the study of patterns of past prices and

other summary statistics about security trading. By extrapolating price trends, technical trading rules usually add a positive feedback to the dynamics of financial markets, and thus may be destabilizing. Basically, they believe that shifts in supply and demand can be detected in charts of market movements. The trend followers are assumed to react to buy-sell signals generated by the difference between the current price and the price trend which is formed as an integral with a distributed delay, representing a MA of the historical prices with an exponential decaying weight over a memory length. Here we consider there's being a definite functional relation between the fundamental and the excess demand. The continuous-time model provides a uniform treatment on various MA rules used in the discrete-time model.

This paper is organized as follows: Section 2 presents a three-dimensional financial model with time delays terms. In Section 3, we analyze the global stability conditions of the fundamental steady state by constructing suitable Lyapunov functional. It is interesting to find the existence of the term of the feedback control, that is to say the new model has very good stability behaviours from the stable cycle or bifurcation. In Section 4, numerical simulations show that the deterministic model is able to characterize long deviations of the market price from its fundamental price and stability behaviours observed in financial markets. It is the same to find that an increase in memory length not only can destabilize the market price, but also can stabilize an otherwise unstable market price, leading to stability switching as the memory length increases. Section 5 concludes the paper.

2. A DYNAMIC FINANCE MODEL

From Brock and Hommes, Chiarella and He, we can understand more about the current HAM framework [9-11], and, in particular, Chiarella, He, and Hommes in discrete-time setup [5]. He at al. have reported an asset pricing model in a continuous-time framework with two different types of heterogeneous traders, fundamentalists and chartists, who trade according to fundamental analysis and technical analysis, respectively [6-7]. The market price is arrived at via a market maker scenario in line with Day and Huang [12] and Chiarella and He [11]. On account of previous work, see from Brock and Hommes (1998), Chiarella and He (2003), Day and Huang (1990), and He et al. (2010), we have the delay differential system below:

$$\begin{cases} \frac{dp(t)}{dt} = \mu[\alpha\beta_f(F(t) - p(t)) + (1 - \alpha) \tanh(\beta_c(\rho p(t) + (1 - \rho)F(t) - u(t)))],\\ \frac{du(t)}{dt} = \frac{k}{1 - e^{-k\tau_1}}[p(t) - e^{-k\tau_1}p(t - \tau_1) - (1 - e^{-k\tau_1})u(t)],\\ \frac{dF(t)}{dt} = \delta[\beta_f(F(t) - p(t)) + \beta_c(p(t) - p(t - \tau_2))]. \end{cases}$$
(2.1)

Considering in a market with a risky asset (such as stock market index), p(t) indicates the (cum dividend) price per share of the risky asset at time t. The fundamentalists are trading based on their estimated fundamental price. They believe that the market price p(t) is mean-reverting to the fundamental price F(t). Let $\rho p(t) + (1 - \rho)F(t)$ be the average price trend of the market price and fundamental value weighted by ρ , where $\rho \in [0,1]$. The fundamental price is closely related with the market excess demand, and is adjusted according to the aggregate excess demand, that is

$$\frac{dF(t)}{dt} = \delta[\beta_f(F(t) - p(t)) + \beta_c(p(t) - p(t - \tau_2))]$$

where δ represents the speed of the fundamental price adjusted according to the excess demand. u(t) is measured by an exponentially decayed weighted average of historical price over a time interval

 $[t-\tau_1]$ with time delay $0 < \tau_1 < \infty$, namely $u(t) = \frac{1}{A} \int_{t-\tau_1}^t e^{-k(t-s)} p(t) ds$, $A = \frac{1-e^{-k\tau_1}}{k}$, where

k > 0 measures the decaying rate of the weights on the historical price and A is a normalization constant. The parameter α represents the proportion of fundamentalists, where $\alpha \in [0,1]$.

3. STABILITY ANALYSIS

It's easy to see that p(t) = u(t) = F(t) is the equilibrium of the system (2.1) where the equilibrium steady state price is given by the fundamental function. We therefore call (p, u, F) = (F, F, F) the fundamental steady state.

In this section, we study the dynamics of the deterministic model (2.1), including the stability and bifurcation of the fundamental steady state. In the following, we provide a result on the global stability of the fundamental steady state by constructing suitable Lyapunov functional.

Theorem 3.1. The fundamental steady state of the system (2.1) is globally asymptotically stable if when either

$$\begin{split} \gamma_{c} - \gamma_{f} > & \frac{3\delta_{c}}{2} and\gamma_{f} + \frac{\gamma_{c}(1-\rho)}{2} + \delta_{c} - \frac{\gamma_{c}\rho}{2} - \frac{3}{2}\delta_{f} < \frac{k(1+e^{-k\tau_{1}})}{2(1-e^{-k\tau_{1}})} < \frac{k}{2(1-e^{-k\tau_{1}})} + \frac{k}{2} - \frac{\delta_{c}}{4} and \frac{k}{1-e^{-k\tau_{1}}} + \delta_{f} < \gamma_{c} + \delta_{c} \\ and & \frac{\gamma_{c} + k}{4} - \frac{\delta_{c} + \delta_{f}}{4} < \frac{ke^{-k\tau_{1}}}{2(1-e^{-k\tau_{1}})} < \gamma_{f} - \gamma_{c}\rho - \delta_{f}, \end{split}$$

or

$$\begin{split} \gamma_{c} - \gamma_{f} < & \frac{3\delta_{c}}{2} and \gamma_{f} + \frac{\gamma_{c}(1-\rho)}{2} + \delta_{c} - \frac{\gamma_{c}\rho}{2} - \frac{3}{2}\delta_{f} < & \frac{k(1+e^{-k\tau_{1}})}{2(1-e^{-k\tau_{1}})} < \frac{k}{2(1-e^{-k\tau_{1}})} + \frac{k}{2} - \frac{\delta_{c}}{4} and \frac{k}{1-e^{-k\tau_{1}}} + \delta_{f} < \gamma_{c} + \delta_{c} \\ and & \frac{\gamma_{c} + k}{4} - \frac{\delta_{c} + \delta_{f}}{4} < & \frac{ke^{-k\tau_{1}}}{2(1-e^{-k\tau_{1}})} < & \frac{\gamma_{f}}{2} + \frac{\gamma_{c}(1-\rho)}{2} - \delta_{f} - \frac{3}{4}\delta_{c}. \end{split}$$

Proof. In the following, we consider $0 < \alpha < 1$ and denote: $\gamma_f = \mu \alpha \beta_f (>0), \quad \gamma_c = \mu (1-\alpha) \beta_c (>0), \quad \delta_f = \delta \beta_f (>0), \quad \delta_c = \delta \beta_c (>0).$ The system becomes

$$\begin{cases} \frac{dx(t)}{dt} = \gamma_f(z(t) - x(t)) + \mu(1 - \alpha) \tanh(\beta_c(\rho x(t) + (1 - \rho)z(t) - y(t))), \\ \frac{dy(t)}{dt} = \frac{k}{1 - e^{-k\tau_1}} [x(t) - e^{-k\tau_1}x(t - \tau_1) - (1 - e^{-k\tau_1})y(t)], \\ \frac{dz(t)}{dt} = \delta_f(z(t) - x(t)) + \delta_c(x(t) - x(t - \tau_2)). \end{cases}$$
(3.3)

Define a Lyapunov functional

$$V(x(t), y(t), z(t), x(t - \tau_1), x(t - \tau_2)) = \frac{1}{2} [(x(t) - z(t))^2 + (y(t) - z(t))^2] + \frac{\delta_c}{2} \int_{t - \tau_2}^t [(x(s) - z(s))^2 + z^2(s)] ds$$

+ $\frac{ke^{-k\tau_1}}{2(1 - e^{-k\tau_1})} \int_{t - \tau_1}^t [(x(s) - z(s))^2 + z^2(s)] ds.$

Then one can verify that

$$\begin{aligned} \frac{dV}{dt} &\leq \left(\gamma_{c}\rho + \delta_{f} + \frac{ke^{-k\tau_{1}}}{2(1 - e^{-k\tau_{1}})} - \gamma_{f}\right)x^{2}(t) + \left(\frac{\delta_{c}}{2} + \frac{ke^{-k\tau_{1}}}{1 - e^{-k\tau_{1}}} - k\right)y^{2}(t) + \left(2\delta_{f} + \frac{3}{2}\delta_{c} + \frac{ke^{-k\tau_{1}}}{1 - e^{-k\tau_{1}}} - \gamma_{f} - \gamma_{c}(1 - \rho)\right)z^{2}(t) \\ &+ \left(\frac{k}{1 - e^{-k\tau_{1}}} + \delta_{f} - \gamma_{c} - \delta_{c}\right)x(t)y(t) + 2\left(\gamma_{f} + \frac{\gamma_{c}(1 - \rho)}{2} + \delta_{c} - \frac{k(1 + e^{-k\tau_{1}})}{2(1 - e^{-k\tau_{1}})} - \frac{\gamma_{c}\rho}{2} - \frac{3\delta_{f}}{2}\right)x(t)z(t) \\ &+ \left(\gamma_{c} + k - \frac{2ke^{-k\tau_{1}}}{1 - e^{-k\tau_{1}}} - \delta_{f} - \delta_{c}\right)y(t)z(t) - \delta_{c}y(t)z(t - \tau_{2}) < 0. \quad (x(t), y(t), z(t) > 0) \end{aligned}$$

There exist M > 0 such that $\sqrt{x^2(t) + y^2(t) + z^2}(t) \ge M$ implies $\frac{dV}{dt} < 0.$

This shows that the solutions of system (3.3) are uniformly ultimately bounded. Applying Lyapunov Theorem, we can show the global stability of the fundamental steady state under the condition (3.2). From the condition obtained above, we cannot get more intuitive instructions to describe the system stability switches. But we could discuss the stability of the original system from the following conversion.

Because p(t) = u(t) = F(t) is the equilibrium of the system (2.1), let P(t) = p(t) - F(t), U(t) = u(t) - F(t), then the system (2.1) becomes

(3.2)

$$\begin{cases} \frac{dP(t)}{dt} = (-\gamma_f + \delta_f)P(t) + \mu(1-\alpha)\tanh(\beta_c(\rho P(t) - U(t))) - \delta_c(P(t) - P(t-\tau_2) - \delta_c(F(t) - F(t-\tau_2)), \\ \frac{dU(t)}{dt} = (\frac{k}{1-e^{-k\tau_1}} + \delta_f)P(t) - \frac{ke^{-k\tau_1}}{1-e^{-k\tau_1}}P(t-\tau_1) - kU(t) + \frac{ke^{-k\tau_1}}{1-e^{-k\tau_1}}[F(t) - F(t-\tau_1)] - \delta_c(P(t) - P(t-\tau_2)), \\ -\delta_c(F(t) - F(t-\tau_2)), \\ \frac{dF(t)}{dt} = (\delta_c - \delta_f)P(t) - \delta_c P(t-\tau_2) + \delta_c(F(t) - F(t-\tau_2)). \end{cases}$$
(3.4)

Here $\delta_c(P(t) - P(t - \tau_2))$ and $\delta_c(F(t) - F(t - \tau_2))$ can be viewed as delayed feedbacks control, and from the control theory perspective, the stability of the system has a more intuitive performance. We know that there are many previous works on the feedback control, but for now, we do not find a specific solution on solving differential equations with several delays. So in the following study, we will use numerical simulation to illustrate the stability switching of the system (3.4).

4. NUMERICAL SIMULATIONS



Fig.1. Time series plots of the market price and the fundamental value.

Here
$$\mu = 1$$
, $\alpha = 0.41$, $\rho = 0.5$, $\beta_c = 1.4$, $\beta_f = 1.4$, $k = 0.06$, $\tau_1 = \tau_2 = 1$, and $\delta = 0.01$.

From the Fig.1, we can see that the development of the price dynamic system. As can be see, price move erratically around their fundamental values. There are periods where price are close to the fundamental value but occasionally large bubbles set in. A prominent example is given around time step 300-400, where the distance between the market prices and the fundamental values implies a substantial overvaluation.

Alessandro Sansone and Giuseppe Garofalo [13] and He and Li (2009, 2010) have analyzed about the effect on stability by changing the proportion of fundamentalists and chartists. In the following, we give some numerical simulation of the system (2.1) with different parameters.



Fig.2. Stability switching of the fundamental steady state with respect to lpha .

Here $\mu = 1$, $\rho = 0.5$, $\beta_c = 1.4$, $\beta_f = 1.4$, k = 0.05, $\tau_1 = 5$, $\tau_2 = 1$, and $\delta = 0.1$.

From the Fig.2, we can see the effect of the fraction of the fundamentalists display in the price volatility. The smaller value of a, the greater volatility do the market prices have around the fundamental valves. That is to say, the chartists play a destabilizing role in market finance system, while the fundamentalists play the part of stabilizing role. In fact, larger departures from fundamental value are needed for the fundamentalists to drive the price toward the fundamental value. With μ =1, ρ = 0.5, β_c = 1.4, β_f =1.4, k = 0.05, τ_1 = 5, τ_2 = 1, and δ = 0.1, the fundamentalists steady state is stable for 0.367 ≤ α < 1, and a stable cycle for 0 < α < 0.367.



Fig.3. Stability switching of the fundamental steady state and the corresponding diagrams with the respect

to
$$\tau$$
 . Here μ =1, ho = 0.5, β_c = 1.4, β_f =1.4, k = 0.05, and δ = 0.1

From the charts above, we can find that the values of τ_2 affect the stability switching of the fundamental steady state. The original system will accelerate to a steady state with the appropriate values of τ_2 . Fig.3 (a)-(b) show that (b) to be faster than (a) plans to achieve steady state. Fig.3 (c)-(d) show that the stability of the fundamental steady state switches at the appropriate values of τ_2 . Here we can understand that τ_2 plays the role of time delay control.

5. CONCLUSIONS

This paper develops a new continuous-time dynamic model when the price trend of the trend followers is formed by geometrically weighted and continuously distributed lagged prices. The HAM with time delays provides a unified treatment to the discrete-time HAMs where the price trend follows weighted moving average rules. In this paper, we have analyzed the global stability of the financial system by constructing Lyapunov function and give some numerical simulations. Through the establishment of a suitable Lyapunov function, we obtain the conditions for the global stability of the financial system; we hope there is also greater improvement in finding a better Lyapunov function. Numerical simulations show that the deterministic model is able to characterize long deviations of the market price from its fundamental price and stability behaviours observed in financial markets. Also there are some difficulties encountered in current work, such as how to find a better Lyapunov function, how to solve the problem with several delays and hope for a better future.

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